SCEGGS Darlinghurst

2009

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\displaystyle\int_0^{\frac{\pi}{3}} \sec^3 x \tan x \; dx$ **2**

(b) Find $\displaystyle\int \sqrt{\frac{5-x}{5+x}} \; dx$. **3**

(c) (i) Find real numbers A, B and C such that **2**

$$\frac{10}{(3+x)\left(1+x^2\right)} \equiv \frac{A}{3+x} + \frac{Bx+C}{1+x^2}.$$

(ii) Use the substitution $t = \tan\theta$ to find $\displaystyle\int \frac{10}{3+\tan\theta} \; d\theta$. **3**

(d) For $n \geq 1$, let $I_n = \displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^n}$.

(i) By writing $\displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^n}$ as $\displaystyle\int_0^1 1 \times \frac{dx}{\left(x^2+1\right)^n}$, and using integration by parts, **3**

show that

$$2nI_{n+1} = 2^{-n} + (2n-1)\, I_n .$$

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{dx}{\left(x^2+1\right)^3}$. **2**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) The complex number w is given by $w = -1 + \sqrt{3}\, i$.

(i) Show that $w^2 = 2\overline{w}$. **1**

(ii) Evaluate $|\, w \,|$ and $\arg w$. **2**

(iii) Show that w is a root of $z^3 - 8 = 0$. **1**

(b) On separate diagrams, draw a neat sketch of the locus defined by

(i) $|\, z - 1 - 3i \,| \le 2$ and $\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{2}$. **2**

(ii) $\arg\left(\dfrac{z-i}{z-1}\right) = \dfrac{\pi}{2}$. **2**

(c) By considering the binomial expansion of $(1+i)^n$ show that **3**

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{\frac{n}{2}} \cos\frac{n\pi}{4}.$$

(d) The points O, I, Z and P on the Argand Plane represent the complex numbers 0, 1, z and $z + 1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1, with $0 < \theta < \pi$.

(i) Explain why $OIPZ$ is a rhombus. **1**

(ii) Show that $\dfrac{z-1}{z+1}$ is purely imaginary. **2**

(iii) Find the modulus of $z + 1$ in terms of θ. **1**

End of Question 2

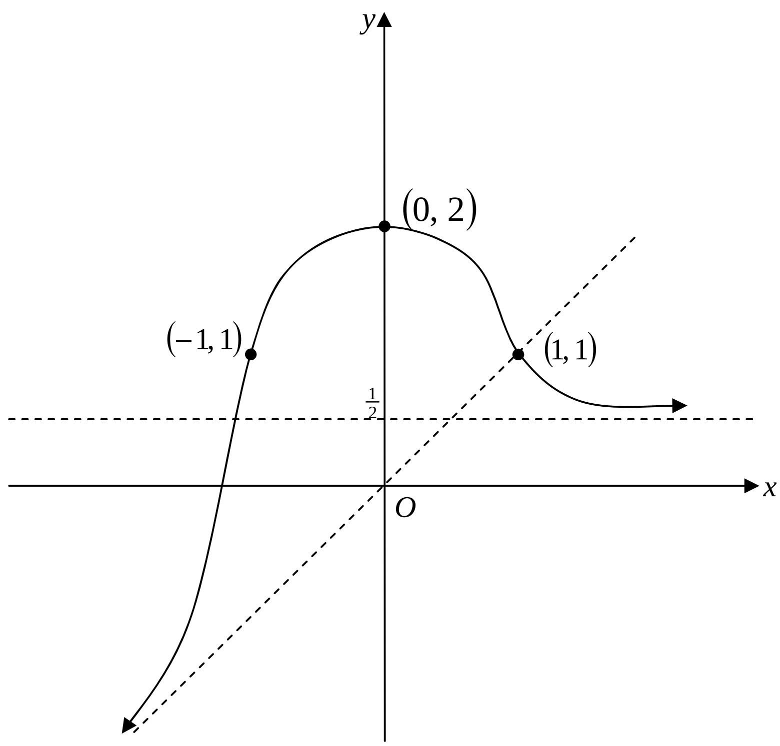
Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The locus defined by $|z-2|-|z+2|=2$ corresponds to part of a hyperbola in the Argand Plane. **3**

Sketch the locus labeling the foci, directrices, asymptotes and any intercepts with the axes.

(b)



The diagram shows the graph of $y = f(x)$. The lines $y = x$ and $y = \frac{1}{2}$ are both asymptotes.

On the answer page provided, draw separate sketches of the following graphs. Clearly indicate any important features.

(i) $y = (f(x))^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

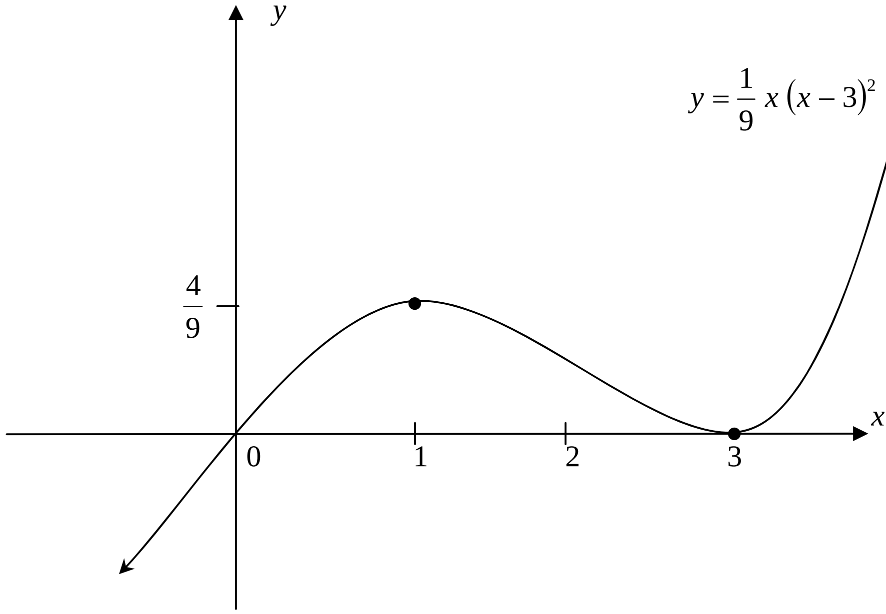
(iii) $y = f(x) - x$ **2**

Question 3 continues on page 5

Marks

Question 3 (continued)

(c)



(i) Given the sketch of $y = \frac{1}{9}x(x-3)^2$ above, sketch the curve $y^2 = \frac{1}{9}x(x-3)^2$. **2**

(ii) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y for $y^2 = \frac{1}{9}x(x-3)^2$. **1**

(iii) Given that the length of a curve between $x = a$ and $x = b$ is given by **3**

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

find the entire length of the curve $y^2 = \frac{1}{9}x(x-3)^2$ for $0 \le x \le 3$.

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$.

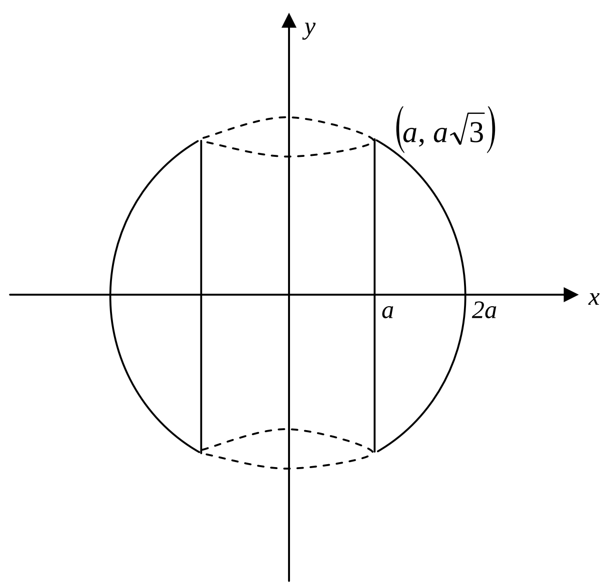
(i) If $P(x)$ has zeroes $a + bi$ and $a + 2bi$, where a and b are real and $b > 0$, find the values of a and b. **3**

(ii) Hence express $P(x)$ as the product of two quadratic factors with real coefficients. **1**

(b) The region bounded by the curve $y = \cos x$ and the coordinate axes is rotated about the y-axis. **3**

Use the method of cylindrical shells to find the volume of the solid formed.

(c) **3**



A cylindrical hole of radius a cm is bored through the centre of a sphere of radius $2a$ cm.

Show that the volume of the remaining solid is $4\sqrt{3}a^3\pi$ cm^3.

Question 4 continues on page 7

Marks

Question 4 (continued)

(d) The cubic equation $x^3 + kx + 1 = 0$, where k is a constant, has roots α, β and γ. For each positive integer n, $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) State the value of S_1 and express S_2 in terms of k. **2**

(ii) Show that for all n, $S_{n+3} + kS_{n+1} + S_n = 0$. **2**

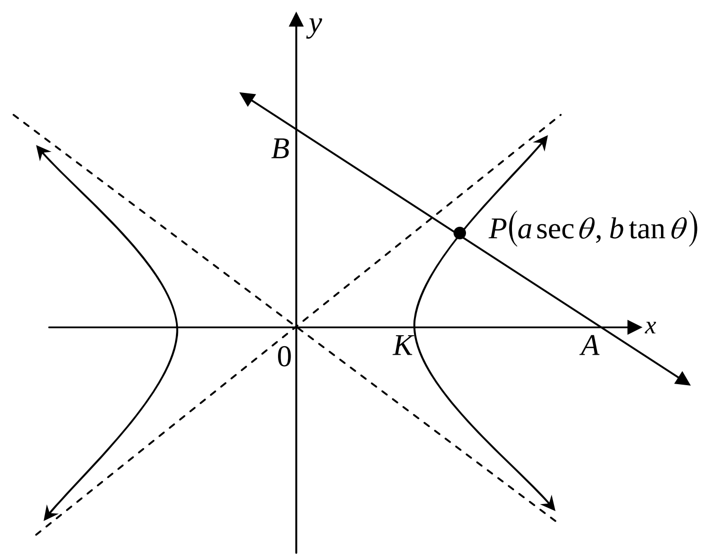
(iii) Hence, or otherwise, express $\alpha^4 + \beta^4 + \gamma^4$ in terms of k. **1**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a)



The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$, cuts the positive x–axis at the point K.

The normal to the hyperbola at the point $P(a\sec\theta, b\tan\theta)$ cuts the x–axis at A and the y–axis at B, as shown in the diagram.

(i) Show that the equation of the normal to the hyperbola at the point P is **2**

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 .$$

(ii) Find the midpoint M of AB. **3**

(iii) Find the point G such that G divides the interval OM in the ratio 2:1. **1**

(iv) Show that the locus of G is a hyperbola and find the point L at which this locus cuts the positive x–axis. **3**

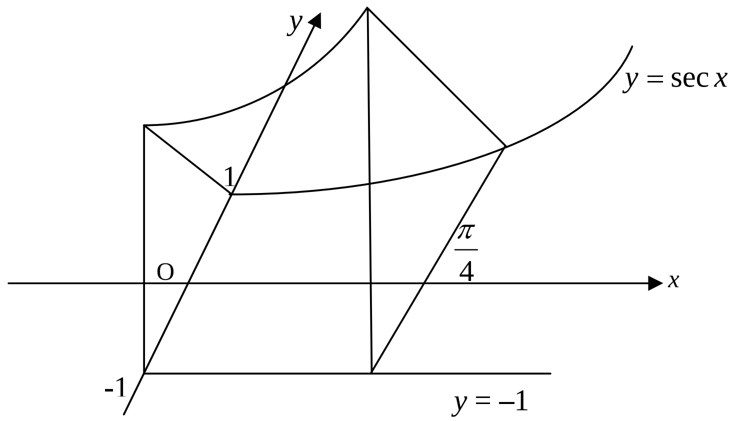
(v) If $\dfrac{OL}{OK} < 1$, show that $1 < e < \sqrt{3}$. **2**

Question 5 continues on page 9

Marks

Question 5 (continued)

(b) The base of a solid is the region in the xy plane enclosed by the curve $y = \sec x$ and $y = -1$ for $0 \le x \le \frac{\pi}{4}$. Each cross-section perpendicular to the x–axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at a distance x from the y – axis is $\frac{\sqrt{3}}{4}(\sec x + 1)^2$. **1**

(ii) Hence find the volume of the solid. **3**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_1 = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$ and let $I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx$

(i) Using a suitable substitution show that $I_1 = I_2$. **1**

(ii) Find the value of $I_1 + I_2$ and hence evaluate $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$. **3**

(b) Let $z = \cos\theta + i \sin\theta$ be any complex number of modulus 1.

(i) Show that $\frac{z^2 - 1}{z} = 2i \sin\theta$. **2**

(ii) Using the formula for the sum of a Geometric Progression and the result in part (i), prove that **2**

$$z + z^3 + z^5 + z^7 + z^9 = \frac{\sin 10\theta + i(1 - \cos 10\theta)}{2 \sin\theta}.$$

(iii) Hence write down a simplified expression for **3**

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta$$

and find the general solution to

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{1}{2}.$$

Question 6 continues on page 11

Marks

Question 6 (continued)

(c) Seven players are entered in a round robin tennis competition. Each round consists of three singles matches with the 7th player obtaining a bye.

In how many ways can the first round of the competition be arranged if

(i) there are no restrictions? **2**

(ii) Amy is not playing Ben? **2**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose $x > 0, \quad y > 0, \quad z > 0$.

(i) Prove that $x^2 + y^2 \geq 2xy$. **1**

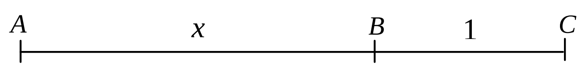
(ii) Hence, or otherwise, prove that $\frac{x}{y} + \frac{y}{z} \geq 2$. **1**

(iii) Prove that $x^3 + y^3 \geq xyz\left(\frac{x}{z} + \frac{y}{z}\right)$. **1**

(iv) Hence show that $x^3 + y^3 + z^3 \geq 3xyz$. **1**

(v) Deduce that $(a+b+c)(a+b+d)(a+c+d)(b+c+d) \geq 81abcd$ where $a > 0, \quad b > 0, \quad c > 0, \quad d > 0$. **1**

(b) (i) **1**



The diagram shows a straight line segment AC divided by B in the ratio $x:1$.

If A divides CB externally in the same ratio that B divides AC internally, show that

$$x^2 = x + 1$$

(ii) A sequence $\{F_n\}$, the Fibonacci numbers, is defined by $F_1 = 1, \quad F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. **3**

The golden ratio φ, and its conjugate root θ, are the positive and negative solutions to the equation in part (i).

Prove by induction, that the closed form expression for the Fibonacci numbers is given by

$$F_n = \frac{\varphi^n - \theta^n}{\sqrt{5}}.$$

Question 7 continues on page 13

Marks

Question 7 (continued)

(c) A projectile is fired vertically upwards with initial speed u. It experiences air resistance proportional to its speed as well as gravitational acceleration g, so that in its upwards flight, the equation of motion is $\ddot{x} = -g - kv$, for some constant $k > 0$ and where v is the velocity of the projectile.

(i) Show that the time T taken to reach its maximum height is given by **3**

$$T = \frac{1}{k}\log_e\left(1 + \frac{ku}{g}\right).$$

(ii) By first writing $\ddot{x}$ as $v\dfrac{dv}{dx}$, show that the maximum height of the particle H is given by **3**

$$H = \frac{u - gT}{k}.$$

End of Question 7

Marks

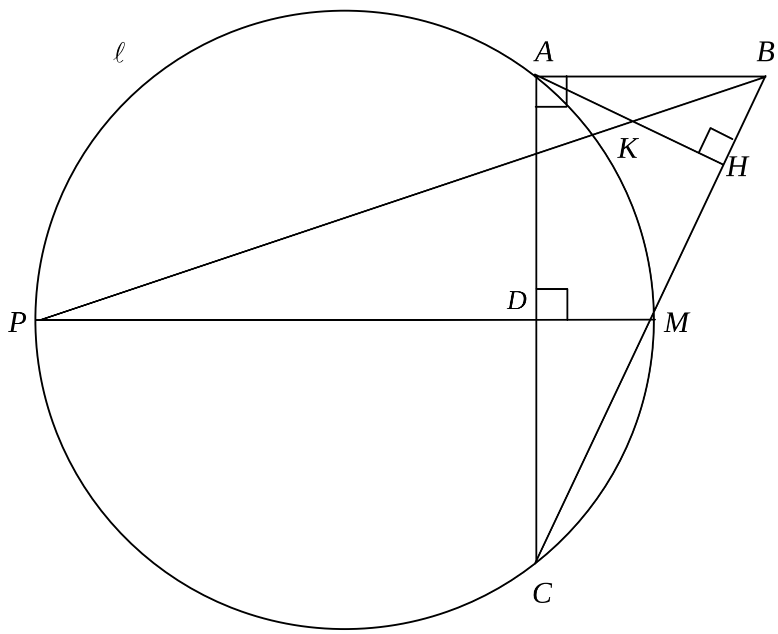
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) α is a double root of the equation $x^n - bx^2 + c = 0$.

(i) Show that $\alpha^2 = \dfrac{nc}{nb - 2b}$. **2**

(ii) Hence show that $n^n c^{n-2} = 4b^n (n-2)^{n-2}$. **2**

(b)



In ΔABC, $\angle A = 90°$, M is the midpoint of BC and H is the foot of the altitude from A to BC. A circle ℓ is drawn through points A, M and C. The line passing through M perpendicular to AC meets AC at D and the circle ℓ again at P. BP intersects AH at K.

(i) Show that PM is the diameter of the circle ℓ. **2**

(ii) Show that $\Delta MCD \mathrel{|||} \Delta MPC$. **2**

(iii) Hence show that $\Delta DMB \mathrel{|||} \Delta BMP$. **2**

(iv) Deduce that $\angle DBM = \angle ABK$. **2**

(v) By making further use of similar triangles, or otherwise, show that $AK = KH$. **3**

End of Paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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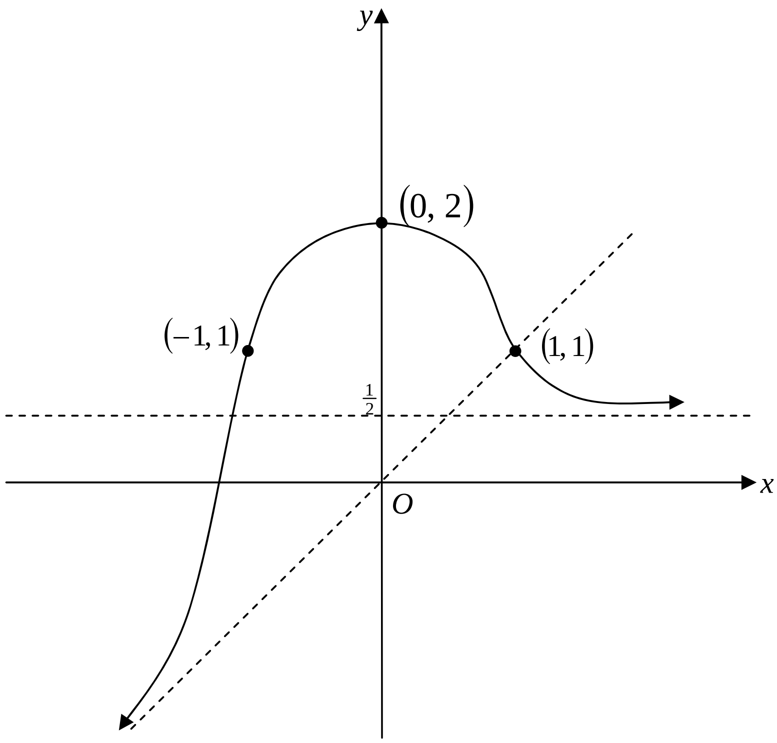
Mathematics Extension 2

Centre Number

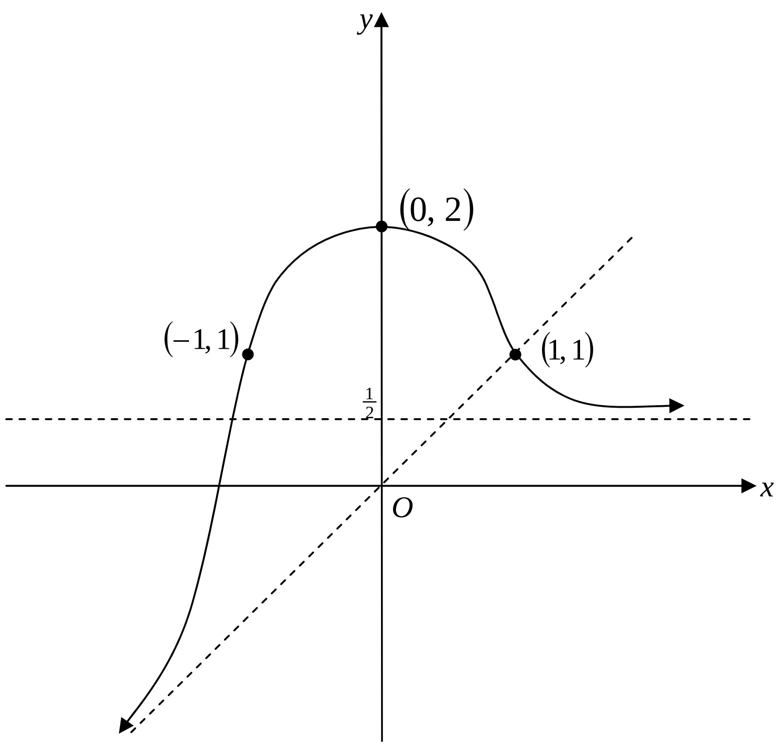
Questions 3 (b)

Student Number

(i)

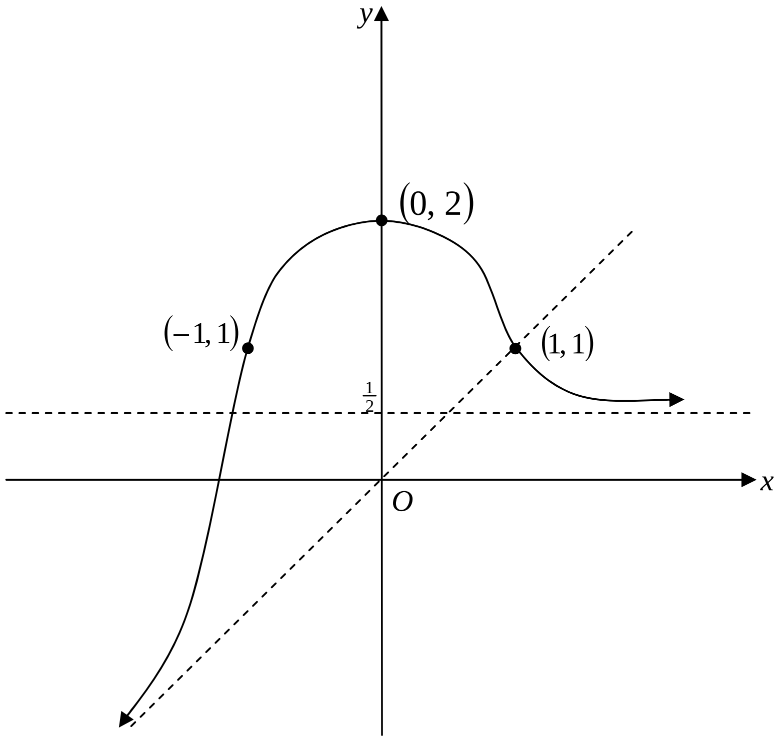


(ii)



Question 3b (continued)

(iii)



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EXT II TRIAL PAPER
SOLUTIONS

Question 1 (15 marks)

Calc 15

Overall, I don't think Q1 was as well done as it should have been. Mistakes ranged from trivial + & - signs to much more fundamental ones. You should be aiming for 15/15 in 15 minutes for this question.

(a) $\int_0^{\pi/3} \sec^3 x \tan x \, dx$

$= \int_0^{\pi/3} \sec^2 x . \sec x \tan x \, dx$

$= \left[\frac{1}{3}\sec^3 x\right]_0^{\pi/3} = \frac{7}{3}$ ✓ ✓

(b) $\int \sqrt{\frac{5-x}{5+x}} \, dx$

rationalise the numerator

$= \int \frac{5-x}{\sqrt{25-x^2}} \, dx$

$= \int \frac{5}{\sqrt{25-x^2}} + \frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}} \, dx$ ✓

$= 5\sin^{-1}\left(\frac{x}{5}\right) + \sqrt{25-x^2} + C$ ✓ ✓

(c) (i) $\frac{10}{(3+x)(1+x^2)} = \frac{\boxed{1}}{3+x} + \frac{\boxed{-1}x + \boxed{3}}{1+x^2}$

$A = 1, \; B = -1, \; C = 3$ ✓✓

(ii) $t = \tan\theta$

$\frac{dt}{d\theta} = \sec^2\theta = 1 + \tan^2\theta$

$d\theta = \frac{dt}{1+t^2}$

$\therefore \int \frac{10}{3+\tan\theta} \, d\theta$

$= \int \frac{10}{(3+t)(1+t^2)} \, dt$ ✓

Working for substitution must be shown.

This question did not require the use of t-formulae!

$$= \int \frac{1}{3+t} + \frac{-t+3}{1+t^2} \, dt$$

$$= \int \frac{1}{3+t} - \frac{1}{2}\int \frac{2t}{1+t^2} + \int \frac{3}{1+t^2} \, dt$$

✓✓

$$= \ln(3+t) - \tfrac{1}{2}\ln(1+t^2) + 3\tan^{-1}(t) + C$$

$$= \ln\left(\frac{3+\tan\theta}{\sqrt{1+\tan^2\theta}}\right) + 3\theta + C$$

$$= \ln(3\cos\theta + \sin\theta) + 3\theta + C$$

In this particular question, no marks were deducted for not changing back to θ, but don't forget it because they usually are (deducted)!

(d) (i) $I_n = \int_0^1 1 \times \frac{dx}{(x^2+1)^n}$

v' u

$$= \left[x \cdot \frac{1}{(x^2+1)^n}\right]_0^1 - \int_0^1 x \cdot \frac{-n \cdot 2x}{(x^2+1)^{n+1}}$$ ✓

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2+1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}} \, dx$$ ✓

$$= 2^{-n} + 2n(I_n - I_{n+1})$$

+1 −1 to not change the question is a technique worth putting in your toolbox.

$$2n I_{n+1} = 2^{-n} + (2n-1) I_n$$ ✓

(ii) $I_1 = \int_0^1 \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_0^1 = \pi/4$ ✓

This part can certainly be completed whether or not you could complete (i).

$$2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \frac{\pi}{4}$$

$$I_2 = \frac{1}{4} + \frac{\pi}{8}$$

$$4I_3 = \frac{1}{4} + 3I_2 = \frac{1}{4} + \frac{3}{4} + \frac{3\pi}{8}$$

$$I_3 = \frac{1}{4} + \frac{3\pi}{32}$$ ✓

Question 2 (15 marks)

Comm 7

(a) $w = -1+\sqrt{3}i$

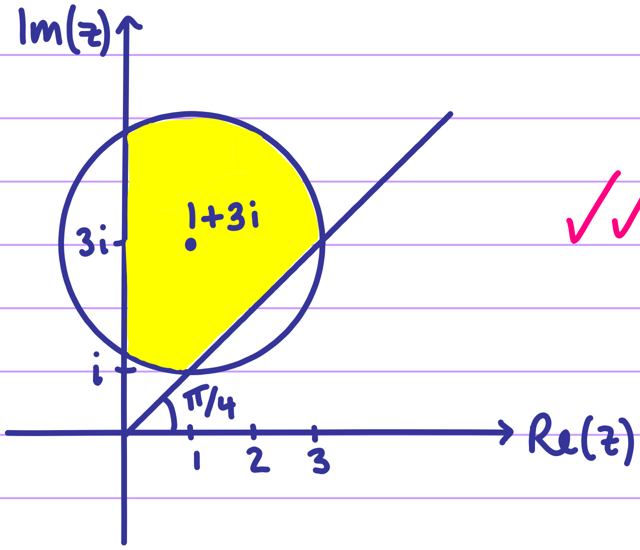
(i) $w^2 = (-1+\sqrt{3}i)^2$
$= 1-2\sqrt{3}i-3$
$= -2-2\sqrt{3}i$
$= 2(-1-\sqrt{3}i)$ ✓
$= 2\bar{w}$ ✓

(ii) $|w| = 2$ ✓
$\arg w = \frac{2\pi}{3}$ ✓

(iii) $w^3 - 8 = \left(2\operatorname{cis}\frac{2\pi}{3}\right)^3 - 8$
$= 2^3 \operatorname{cis} 2\pi - 8$
$= 8\times 1 - 8$
$= 0$ ✓
$\therefore$ w is a root of $z^3-8=0$

While there was the option of finding all three roots & showing w was one of them (using mod-arg or cartesian form), this was not the fastest approach!

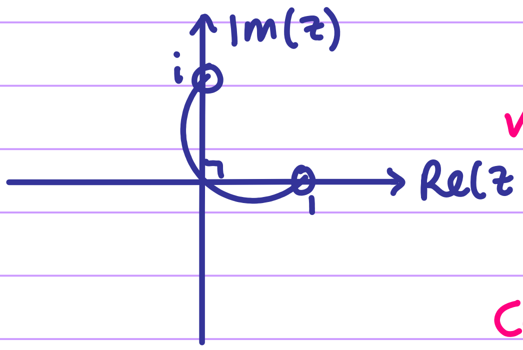
(b) (i)



✓✓

You must draw these graphs as carefully & as accurately & as to scale as you can.
If you can't draw a circle invest in a compass & whether or not you think you can draw a straight line use a ruler!

(ii) $\arg\left(\frac{z-i}{z-1}\right) = \frac{\pi}{2}$



✓✓

Comm 4

Notice the open circles at i & 1 and that the locus passes through the origin.

(c) $(1+i)^n = \binom{n}{0} + \binom{n}{1} i + \binom{n}{2} i^2 + \binom{n}{3} i^3 + \binom{n}{4} i^4 + \cdots$

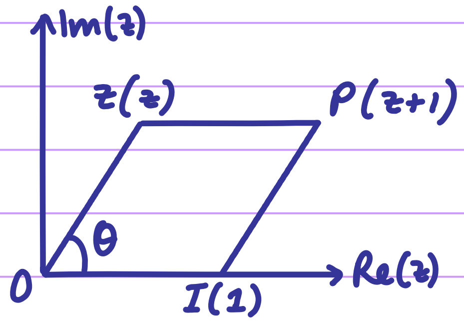
$\left(\sqrt{2} \operatorname{cis}\frac{\pi}{4}\right)^n = \left[\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right] + i\left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right]$

$2^{n/2} \operatorname{cis}\frac{n\pi}{4} = \left[\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right] + i\left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right]$ ✓✓

Taking the real part of both sides

$2^{n/2} \cos\frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots$ ✓

(d)



(i) $\vec{OP} = z + 1 = \vec{OI} + \vec{OZ}$

$\therefore$ OIPZ is a parallelogram

Since $|OZ| = |OI| = 1$, the adjacent sides of OIPZ are equal & thus OIPZ is a rhombus. ✓

Not many could clearly explain this obvious enough fact.

Comm 1

(ii) $\arg\left(\frac{z-1}{z+1}\right)$ = angle of rotation from $(z+1)$ to $(z-1)$

= angle of rotation from $\vec{OP}$ to $\vec{IZ}$

$= \pi/2$ (diagonals of a rhombus are ⊥)

$\therefore \frac{z-1}{z+1}$ is purely imaginary ✓✓

You have to explain why $\arg\left(\frac{z-1}{z+1}\right)$ is the angle between the diagonals.

Comm 2

(iii) $|z+1| = \sqrt{1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(\pi - \theta)}$

$= \sqrt{2 + 2\cos\theta}$ [cos rule] ✓

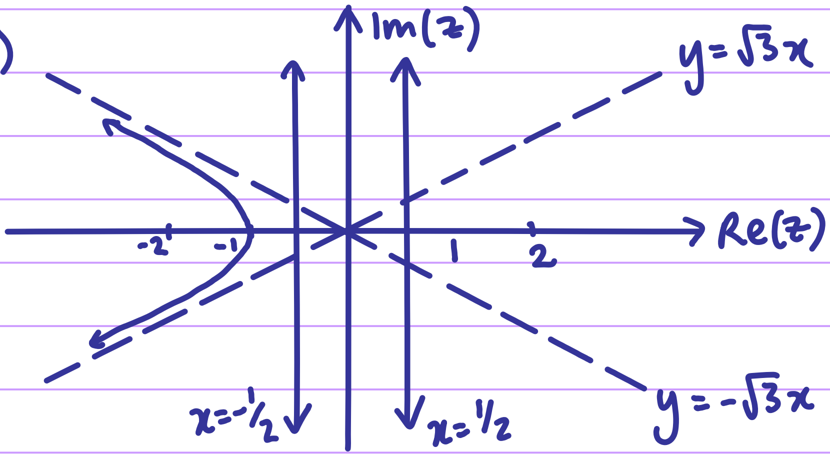
OR $\cos\theta/2 = \frac{|z+1|/2}{1}$

$|z+1| = 2\cos\theta/2$ [SOHCAHTOA]

Question 3 (15 marks)

Calc 4, Comm 11

(a)



This is the easiest version of this question you could possibly get — and it wasn't too successful. The question even told you it was only a branch of the hyperbola!

Foci : $(\pm 2, 0) \Rightarrow ae = 2$

$2a = 2 \Rightarrow a = 1$

$\therefore e = 2$ & $b^2 = a^2(e^2-1)$

$b = \sqrt{3}$

Directrices : $x = \pm 1/2$

Asymptotes : $y = \pm\sqrt{3}x$

x-intercepts : $x = \pm 1$

Note : $|z-2| - |z+2| = 2$

$\Rightarrow$ distance from z to 2 is greater than the distance from z to -2.

✓ correct branch

✓✓ correct features

Comm 3

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Mathematics Extension 2

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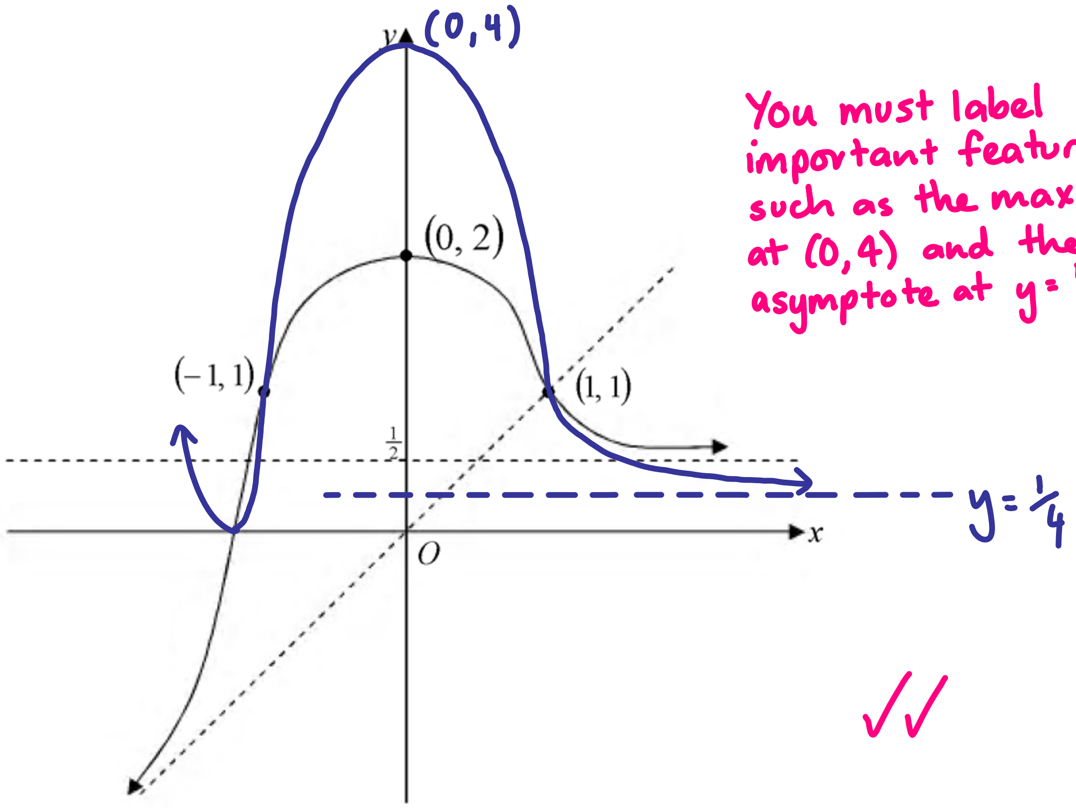
Questions 3 (b)

Student Number

(i)

$y = (f(x))^2$

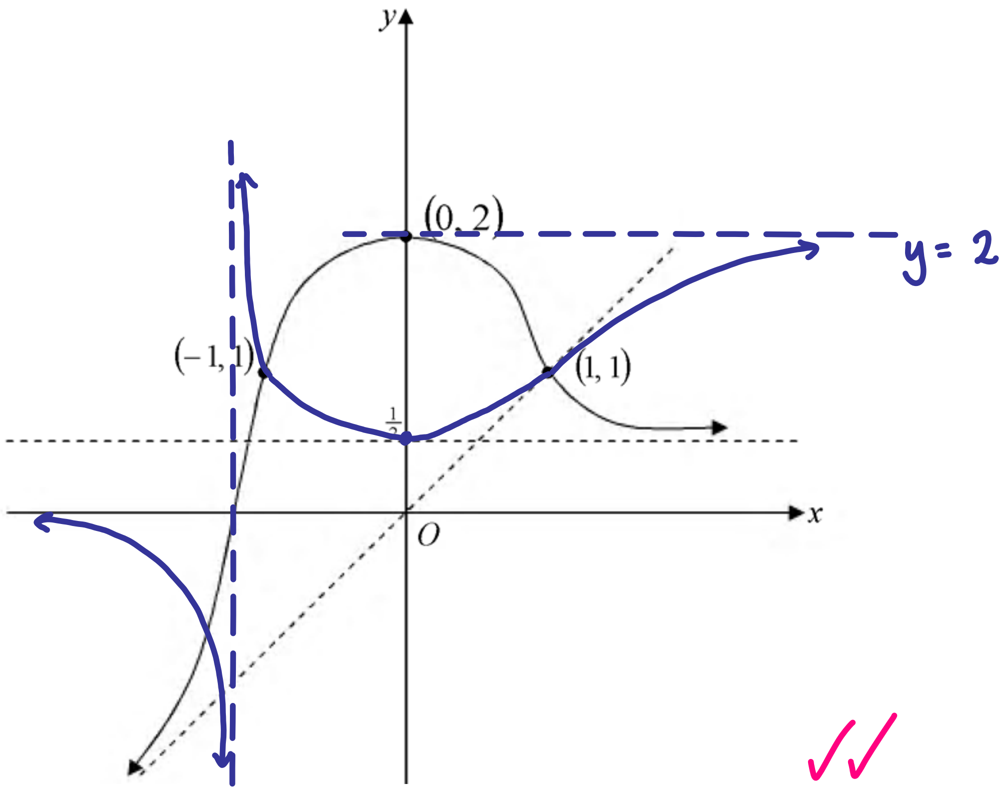


You must label important features such as the max at (0, 4) and the asymptote at $y = 1/4$

✓✓

(ii)

$y = \frac{1}{f(x)}$

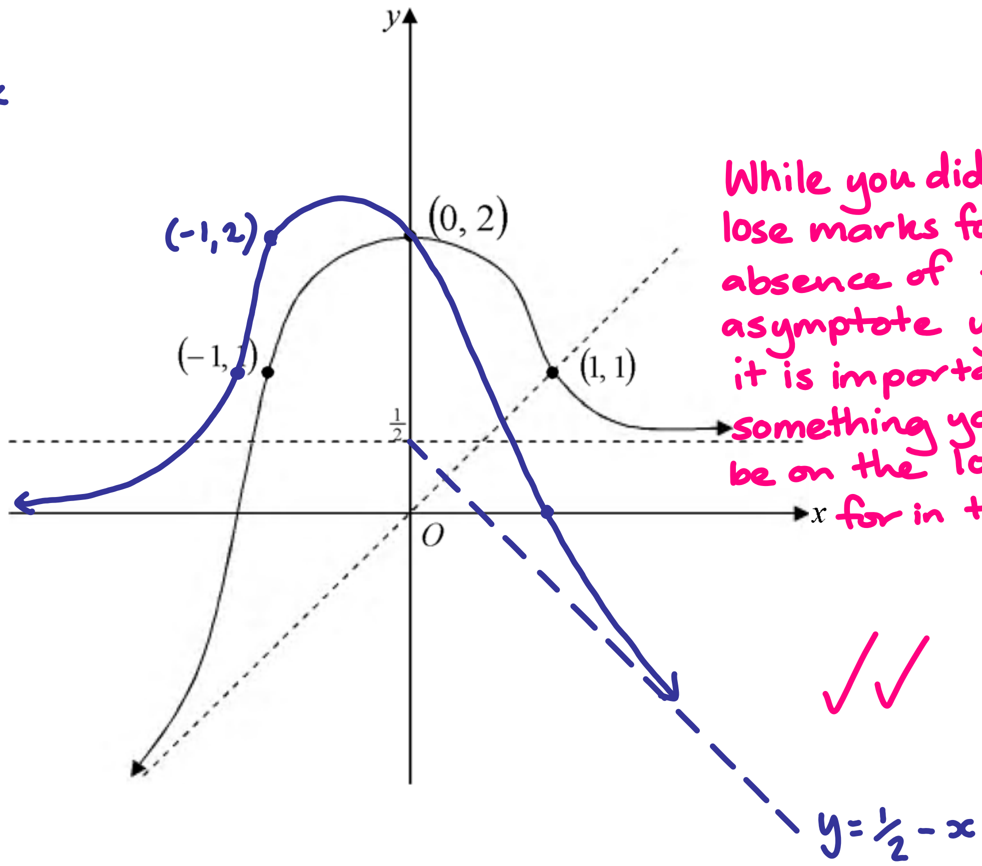


✓✓

Question 3b (continued)

(iii)

$y = f(x) - x$

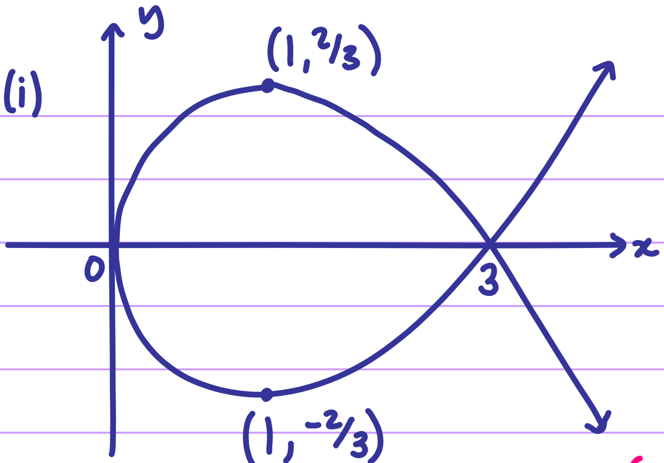


While you didn't lose marks for the absence of the asymptote $y = \frac{1}{2} - x$, it is important & something you should be on the look out for in the future.

✓✓

Comm 6

(c) (i)



✓✓ Comm 2

Important features:

- the vertical asymptote at the origin
- the curve intersects itself with a non-zero gradient at (3,0)
- max/min at $(1, \pm 2/3)$

(ii)

$$y^2 = \frac{1}{9}x(x-3)^2$$

$$= \frac{1}{9}(x^3 - 6x^2 + 9x)$$

$$2y\frac{dy}{dx} = \frac{1}{9}(3x^2 - 12x + 9)$$

$$= \frac{1}{3}(x^2 - 4x + 3)$$

$$\frac{dy}{dx} = \frac{(x-1)(x-3)}{6y}$$ ✓ Calc 1

(iii) length of the top half

$$= \int_0^3 \sqrt{1 + \frac{(x-1)^2(x-3)^2}{36y^2}}\,dx$$

$$= \int_0^3 \sqrt{1 + \frac{(x-1)^2(x-3)^2}{4x(x-3)^2}}\,dx$$ ✓

$$= \int_0^3 \sqrt{\frac{4x + x^2 - 2x + 1}{4x}}\,dx$$

$$= \int_0^3 \frac{x+1}{2\sqrt{x}}\,dx$$ ✓

$$= \left[\frac{x^{3/2}}{3} + x^{1/2}\right]_0^3$$

$$= 2\sqrt{3}$$ ✓ Calc 3

$\therefore$ total length of the curve is $4\sqrt{3}$ units.

Not many had the faith that the algebra/integral would work out. Sometimes you just need to be confident in your own algebra & believe that eventually the integral will be doable.

Full marks were given for an answer of $2\sqrt{3}$, but in fact to get the total length you need to double this answer.

Question 4 (15 marks)

Calc 6

(a) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

(i) Since $P(x)$ has real coefficients, if $a+bi$ & $a+2bi$ are roots, the conjugates $a-bi$ & $a-2bi$ are also roots. ✓

You must state this theorem whenever you need to use it.

Sum of roots $= -b/a$

$a+bi + a-bi + a+2bi + a-2bi = 4$

$4a = 4$

$a = 1$ ✓

Product of roots $= +e/a$

$(a+bi)(a-bi)(a+2bi)(a-2bi) = 10$

$(a^2+b^2)(a^2+4b^2) = 10$

$(1+b^2)(1+4b^2) = 10$

$4b^4 + 5b^2 - 9 = 0$

$(4b^2+9)(b^2-1) = 0$ ✓

$b = 1$ since $b \in \mathbb{R}$ & $b > 0$ ← you really do need to state exactly why $b \neq \pm\sqrt{\frac{-9}{4}}, -1$

(ii) The roots are: $1 \pm i$, $1 \pm 2i$

$1+i, 1-i \longrightarrow$ sum $= 2$, prod $= 2$

$1+2i, 1-2i \longrightarrow$ sum $= 2$, prod $= 5$

$\therefore P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

$= (x^2 - 2x + 2)(x^2 - 2x + 5)$ ✓

(b) $V = \int_0^{\pi/2} 2\pi x \cos x \, dx$ ✓

u dv

$= 2\pi \left(\left[x . \sin x \right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx \right)$ ✓

$= 2\pi \left(\frac{\pi}{2} - 0 + \left[\cos x \right]_0^{\pi/2} \right)$

$= 2\pi \left(\frac{\pi}{2} - 1 \right)$

$= \pi^2 - 2\pi$ units^3. ✓ Calc 3

(c) Taking slices perpendicular to the axis:

$V = 2\pi \int_0^{a\sqrt{3}} x^2 - a^2 \, dy$ ✓

$= 2\pi \int_0^{a\sqrt{3}} 4a^2 - y^2 - a^2 \, dy$

$= 2\pi \left[3a^2 y - \frac{y^3}{3} \right]_0^{a\sqrt{3}}$ ✓

$= 2\pi \left[3a^3\sqrt{3} - a^3\sqrt{3} - 0 \right]$

$= 4\pi a^3 \sqrt{3}$ cm^3. ✓

Calc 3

This question could have been done by shells aswell. Too many got caught up in wanting to subtract a volume from $\frac{4}{3}\pi(2a)^3$ – which just made it harder.

(d) $x^3 + kx + 1 = 0$

(i) $S_1 = \alpha + \beta + \gamma$
$= 0$ ✓

$S_2 = \alpha^2 + \beta^2 + \gamma^2$
$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$
$= 0^2 - 2 \times k$
$= -2k$ ✓

(ii) $S_{n+3} + kS_{n+1} + S_n$

Not everything is induction!

$= (\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3})$
$+ k(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1})$
$+ (\alpha^n + \beta^n + \gamma^n)$ ✓

$= \alpha^n(\alpha^3 + k\alpha + 1) + \beta^n(\beta^3 + k\beta + 1)$
$+ \gamma^n(\gamma^3 + k\gamma + 1)$

$= \alpha^n \times 0 + \beta^n \times 0 + \gamma^n \times 0$
$= 0$ (since α, β, γ are roots & satisfy the equation) ✓

(iii) $S_4 + kS_2 + S_1 = 0$
$S_4 = -S_1 - kS_2$
$= -0 - k \times -2k$
$= 2k^2$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = 2k^2$ ✓

Question 5 (15 marks)

Calc 6, Reas 2

(a) (i) P: $x = a\sec\theta$ $y = b\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\frac{dy}{d\theta} = b\sec^2\theta$

$\Rightarrow \frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$

$\Rightarrow m_N = -\frac{a\tan\theta}{b\sec\theta}$ ✓

∴ EQN OF NORMAL:

$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$\frac{by}{\tan\theta} - b^2 = -\frac{ax}{\sec\theta} + a^2$

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ ✓

Calc 2

(ii) $A: \left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$ ✓

$B: \left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$ ✓

$\therefore M = \left(\frac{(a^2+b^2)\sec\theta}{2a}, \frac{(a^2+b^2)\tan\theta}{2b}\right)$ ✓

(iii) $G = \left(\frac{(a^2+b^2)\sec\theta}{3a}, \frac{(a^2+b^2)\tan\theta}{3b}\right)$ ✓

Part (iii) was an easy application of a formula & there was an abnormally high number who couldn't get it right.

(iv) Locus of G:

$$x = \frac{(a^2+b^2)\sec\theta}{3a} \Rightarrow \sec\theta = \frac{3ax}{a^2+b^2}$$

$$y = \frac{(a^2+b^2)\tan\theta}{3b} \Rightarrow \tan\theta = \frac{3by}{a^2+b^2}$$

To find the cartesian equation you need to get rid of the parameter. ✓

Since $\sec^2\theta - \tan^2\theta = 1$

$$\frac{9a^2x^2}{(a^2+b^2)^2} - \frac{9b^2y^2}{(a^2+b^2)^2} = 1 \quad ✓$$

$$\frac{x^2}{\left(\frac{a^2+b^2}{3a}\right)^2} - \frac{y^2}{\left(\frac{a^2+b^2}{3b}\right)^2} = 1$$

Which is a hyperbola which intersects the x axis at

$$L = \left(\frac{a^2+b^2}{3a}, 0\right) \quad ✓$$

(v) $\frac{OL}{OK} < 1 \Rightarrow \frac{\frac{a^2+b^2}{3a}}{a} < 1$

$\Rightarrow a^2+b^2 < 3a^2$

$\Rightarrow b^2 < 2a^2$

$\Rightarrow a^2(e^2-1) < 2a^2$

$\Rightarrow e^2 - 1 < 2$

$\Rightarrow e^2 < 3$ ✓

$\Rightarrow e < \sqrt{3} \quad (e>0)$

Don't be scared of inequalities. Just start with what you're given & aim to get to the end.

Also, since it's a hyperbola $e>1$

[or $b^2 = a^2(e^2-1) \Rightarrow e^2 = \frac{b^2}{a^2} + 1 > 1 \Rightarrow e > 1$]

$\therefore 1 < e < \sqrt{3}$ ✓

Reas 2

(b) (i) Area $= \frac{1}{2}(y+1)^2 \sin \pi/3$

$= \frac{1}{2}(\sec x + 1)^2 \times \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{4}(\sec x + 1)^2$. ✓

(ii) $V = \int_0^{\pi/4} \frac{\sqrt{3}}{4}(\sec x + 1)^2 \, dx$

$= \frac{\sqrt{3}}{4}\int_0^{\pi/4} \sec^2 x + 2\sec x + 1 \, dx$

$= \frac{\sqrt{3}}{4}\left[\tan x + 2\ln(\sec x + \tan x) + x\right]_0^{\pi/4}$ ✓✓

$= \frac{\sqrt{3}}{4}\left[1 + 2\ln(\sqrt{2}+1) + \frac{\pi}{4} - 0\right]$

$= \frac{\sqrt{3}}{4}\left[1 + 2\ln(\sqrt{2}+1) + \frac{\pi}{4}\right]$ ✓ units^3

Calc 4

Question 6 (15 marks)

Calc 4, Reas 11

(a) (i) let $x = \pi - u$ $\quad$ $x = 0,\ u = \pi$

$dx = -du$ $\quad$ $x = \pi,\ u = 0$

$$I_1 = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$$

$$= \int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1 + \cos^2(\pi - u)} \cdot -du$$

$$= \int_0^{\pi} \frac{(\pi - u)\sin u}{1 + \cos^2 u}\, du \quad ✓$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx = I_2$$

When doing a substitution you need to substitute EVERYTHING — the limits & dx too!

(ii) $$I_1 + I_2 = \int_0^{\pi} \frac{x \sin x + (\pi - x)\sin x}{1 + \cos^2 x}\, dx$$

$$= -\pi \int_0^{\pi} \frac{(-\sin x)}{1 + \cos^2 x}\, dx$$

← Not many recognised this reverse chain rule.

$$= -\pi \left[\tan^{-1}(\cos x) \right]_0^{\pi} \quad ✓$$

$$= -\pi \left(\tan^{-1}(-1) - \tan^{-1}(1) \right)$$

$$= -\pi \left(-\frac{\pi}{4} - \frac{\pi}{4} \right)$$

$$= \frac{\pi^2}{2} \quad ✓$$

$$\therefore\ I_1 = I_2 = \frac{\pi^2}{4}$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi^2}{4} \quad ✓$$

Calc 4

(b) (i) $\frac{z^2-1}{z} = z - z^{-1}$

$= \text{cis}\,\theta - \text{cis}(-\theta)$ ✓

$= \cos\theta + i\sin\theta - (\cos(-\theta) + i\sin(-\theta))$

$= \cos\theta + i\sin\theta - \cos\theta - -i\sin\theta$

$= 2i\sin\theta$ ✓

OR/

$$\frac{z^2-1}{z} = \frac{(\cos\theta+i\sin\theta)^2-1}{\cos\theta+i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$ ✓

$$= \frac{(\cos^2\theta+\sin^2\theta)(\cos\theta+i\sin\theta) - (\cos\theta - i\sin\theta)}{\cos^2\theta+\sin^2\theta}$$

$= \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$

$= 2i\sin\theta$ ✓

(ii) $z + z^3 + z^5 + z^7 + z^9$

$$= \frac{z\left((z^2)^5 - 1\right)}{z^2-1}$$ ✓

$$= \frac{z^{10}-1}{\frac{z^2-1}{z}}$$

$$= \frac{\cos 10\theta + i\sin 10\theta - 1}{2i\sin\theta} \times \frac{-i}{-i}$$

$$= \frac{\sin 10\theta + i(1-\cos 10\theta)}{2\sin\theta}$$ ✓

(iii) $\text{Re}\left(z+z^3+z^5+z^7+z^9\right)$

$$= \text{Re}\left(\frac{\sin 10\theta + i(1-\cos 10\theta)}{2\sin\theta}\right)$$

$\Rightarrow \cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta$

$$= \frac{\sin 10\theta}{2\sin\theta}$$ ✓

Each part of this question was very clear & followed directly from the previous part. You need to be confident following the lead & direction given in such questions.

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta + \cos 9\theta = \frac{1}{2}$$

$$\Rightarrow \frac{\sin 10\theta}{2\sin\theta} = \frac{1}{2}$$

$$\Rightarrow \sin 10\theta = \sin\theta$$

$$10\theta = \theta + 2\pi k, \quad 10\theta = \pi - \theta + 2\pi k$$

$$9\theta = 2\pi k, \quad 11\theta = \pi + 2\pi k$$

$$\theta = \frac{2\pi k}{9}, \quad \frac{\pi + 2\pi k}{11} \quad k \in \mathbb{Z}$$

✓✓

Reas 7

These type of General Solution questions are absolutely standard & there is no excuse for not knowing this work.

(c) (i)

ways to choose the pairs of players for each of the 3 matches

$$\frac{\binom{7}{2}\binom{5}{2}\binom{3}{2}}{3!} \times \binom{1}{1} = 105$$

✓ ✓

last player has a bye

overcounted by 3! since choosing AvB, CvD, EvF is equivalent to AvB, EvF, CvD etc.

This perms & combs question was quite successful for those who attempted it. Don't be terrified of perms & combs – they are very doable & practice does make perfect.

(ii) Case 1 : Amy or Ben has a bye

choose who has a bye

$$2 \times \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 30$$

Case 2 : Amy & Ben both play

Amy's opponent, Ben's opponent, other 2 to play, person with bye

$$5 \times 4 \times \binom{3}{2} \times \binom{1}{1} = 60$$

∴ Total # ways = 90 ✓✓

Reas 4

OR, # ways Amy plays Ben

$$= \frac{\binom{5}{2}\binom{3}{2}}{2!} \times \binom{1}{1} = 15$$

∴ # ways Amy is not playing Ben = 105 − 15

= 90 ✓✓

Question 7 (15 marks)

Calc 6, Reas 9

(a) (i) $(x-y)^2 \geq 0$

$x^2+y^2-2xy \geq 0$

$x^2+y^2 \geq 2xy$ ✓

(ii) Dividing by xy

$\Rightarrow \frac{x}{y}+\frac{y}{x} \geq 2$ (since $x,y>0$) ✓

(iii) $x^3+y^3 = (x+y)(x^2-xy+y^2)$

$\geq (x+y)(2xy-xy)$

$= xy(x+y)$

$= xyz\left(\frac{x}{z}+\frac{y}{z}\right)$ ✓

(iv) Similarly, $y^3+z^3 \geq xyz\left(\frac{y}{x}+\frac{z}{x}\right)$

& $z^3+x^3 \geq xyz\left(\frac{z}{y}+\frac{x}{y}\right)$

Remember the technique of applying an inequality several times over & then putting it together.

Adding these:

$2(x^3+y^3+z^3) \geq xyz\left(\frac{x}{z}+\frac{y}{z}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}\right)$

$\geq xyz(2+2+2) = 6xyz$

$\therefore x^3+y^3+z^3 \geq 3xyz$ ✓

(v) $x=\sqrt[3]{a}, y=\sqrt[3]{b}, z=\sqrt[3]{c} \Rightarrow a+b+c \geq 3\sqrt[3]{abc}$

Similarly $a+b+d \geq 3\sqrt[3]{abd}$

$a+c+d \geq 3\sqrt[3]{acd}$

$b+c+d \geq 3\sqrt[3]{bcd}$

Multiplying these ✓

$\Rightarrow (a+b+c)(a+b+d)(a+c+d)(b+c+d) \geq 81abcd$

Reas 5

(b) (i) A divides CB externally in the ratio $CA : AB = \varphi + 1 : \varphi$

B divides AC internally in the ratio $AB : BC = \varphi : 1$

$$\therefore \frac{\varphi + 1}{\varphi} = \frac{\varphi}{1}$$

$$\varphi^2 = \varphi + 1$$ ✓

I really liked this question but not many gave it a go which is a pity.

(ii) Solutions to $\varphi^2 = \varphi + 1$

$$\varphi^2 - \varphi - 1 = 0$$

$$\varphi = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \varphi = \frac{1+\sqrt{5}}{2}, \quad \theta = \frac{1-\sqrt{5}}{2}$$

When $n = 1$: $LHS = F_1 = 1$

$$RHS = \frac{\varphi - \theta}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

& when $n = 2$: $LHS = F_2 = 1$

$$RHS = \frac{\varphi^2 - \theta^2}{\sqrt{5}}$$

$$= \frac{(\varphi + 1) - (\theta + 1)}{\sqrt{5}}$$ ✓

$$= \frac{\varphi - \theta}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$$

It's true for $n=1$ & $n=2$ so let $k-1$ & k be integers for which it's true i.e. $F_{k-1} = \frac{\varphi^{k-1} - \theta^{k-1}}{\sqrt{5}}$

$$\& \ F_k = \frac{\varphi^k - \theta^k}{\sqrt{5}}$$

Then $F_{k+1} = F_k + F_{k-1}$

$$= \frac{\varphi^k - \theta^k}{\sqrt{5}} + \frac{\varphi^{k-1} - \theta^{k-1}}{\sqrt{5}}$$

(from assumption)

$$= \frac{\varphi^{k-1}(\varphi+1) - \theta^{k-1}(\theta+1)}{\sqrt{5}}$$

$$= \frac{\varphi^{k-1} \cdot \varphi^2 - \theta^{k-1} \cdot \theta^2}{\sqrt{5}}$$

(since $\varphi^2 = \varphi + 1$ & $\theta^2 = \theta + 1$)

$$= \frac{\varphi^{k+1} - \theta^{k+1}}{\sqrt{5}}$$

& so it's true for the next integer $k+1$

$\therefore$ By strong induction, $F_n = \dfrac{\varphi^n - \theta^{n-1}}{\sqrt{5}}$

(c) ⇑ $\downarrow g+kv$ $t=0$, $v=u$

$\ddot{x} = -g-kv$

This standard question was well done by most.

(i) $\frac{dv}{dt} = -(g+kv)$

$$\int_u^0 \frac{dv}{g+kv} = \int_0^T -dt$$

$$\left[\frac{1}{k}\ln(g+kv)\right]_u^0 = \left[-t\right]_0^T \checkmark\checkmark$$

(2nd ✓ for finding C if an indefinite integral was done, then 3rd ✓ for v=0 & successfully finding T)

$$\frac{1}{k}\ln g - \frac{1}{k}\ln(g+ku) = -T+0$$

$$T = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$$

$$T = \frac{1}{k}\ln\left(1+\frac{ku}{g}\right) \checkmark$$

(ii) $v\frac{dv}{dx} = -(g+kv)$

$$\int_u^0 \frac{v\,dv}{g+kv} = \int_0^H -dx$$

$$\frac{1}{k}\int_u^0 \frac{g+kv}{g+kv} - \frac{g}{g+kv}\,dv = \int_0^H -dx$$

$$\frac{1}{k}\left[v - \frac{g}{k}\ln(g+kv)\right]_u^0 = \left[-x\right]_0^H \checkmark\checkmark$$

(2nd ✓ for finding C if an indefinite integral was done, then 3rd ✓ for v=0 & successfully finding H)

$$\frac{1}{k}\left(0-\frac{g}{k}\ln g - u + \frac{g}{k}\ln(g+ku)\right) = -H$$

$$H = \frac{1}{k}\left(u - g\left(\frac{1}{k}\ln(g+ku) - \frac{1}{k}\ln g\right)\right)$$

$$= \frac{1}{k}\left(u - g\cdot\frac{1}{k}\ln\left(1+\frac{ku}{g}\right)\right) = \frac{u-gT}{k} \checkmark$$

Question 8 (15 marks)

Reas 15

(a)(i) If α is a double root of $x^n - bx^2 + c = 0$
then $\alpha^n - b\alpha^2 + c = 0$ ①

& α is also a single root of $nx^{n-1} - 2bx = 0$
$\Rightarrow n\alpha^{n-1} - 2b\alpha = 0$ ② ✓

② $\times \alpha \Rightarrow n\alpha^n - 2b\alpha^2 = 0$
$\alpha^n = \dfrac{2b\alpha^2}{n}$ ③

substitute into ①
$$\Rightarrow \frac{2b\alpha^2}{n} - b\alpha^2 + c = 0$$
$$\alpha^2\left(\frac{2b - nb}{n}\right) = -c$$
$$\alpha^2 = \frac{nc}{nb - 2b}$$ ✓

(ii) Substituting this into ③
$$\Rightarrow \left(\frac{nc}{nb-2b}\right)^{n/2} = \frac{2b}{n} \cdot \frac{nc}{nb-2b}$$ ✓
$$\Rightarrow n^{n/2} c^{n/2} . (n-2) = 2c . (n-2)^{n/2} b^{n/2}$$

squaring both sides
$$\Rightarrow n^n c^n (n-2)^2 = 4c^2 (n-2)^n b^n$$
$$\Rightarrow n^n c^{n-2} = 4b^n (n-2)^{n-2}$$ ✓

Many got an easy first mark – but some didn't because of trivial differentiating errors which is a pity! You've got to be able to maintain accuracy even in a rush.

The rest of this question was barely attempted.

(b) (i) $AB \parallel DM$ (both lines are $\perp$ to AC)
$\therefore BM = MC \Rightarrow AD = DC$
(intercepts on parallel lines AB & DM are in the same ratio) ✓

$\therefore PM$ bisects AC at right angles & hence PM is the diameter
(the perpendicular bisector of a chord passes through the centre) ✓

(ii) In ΔMCD & ΔMPC
$\angle MDC = 90° = \angle MCP$
(given $MD \perp AC$ & angles in a semicircle are $90°$)
$\angle CMD = \angle PMC$ (common) ✓✓

$\therefore \Delta MCD \parallel\!| \Delta MPC$ (AA similarity test)

Well done to those who recognised this was an easy 2 marks to pick up.

(iii) $\dfrac{MD}{MC} = \dfrac{MC}{MP}$ (corresponding sides in $\parallel\!|$ Δs in the same ratio)

$\Rightarrow \dfrac{MD}{MB} = \dfrac{MB}{MP}$ (since $MC = MB$) ✓

In ΔDMB & ΔBMP
$\dfrac{MD}{MB} = \dfrac{MB}{MP}$ (above)
$\angle DMB = \angle BMP$ (common) ✓

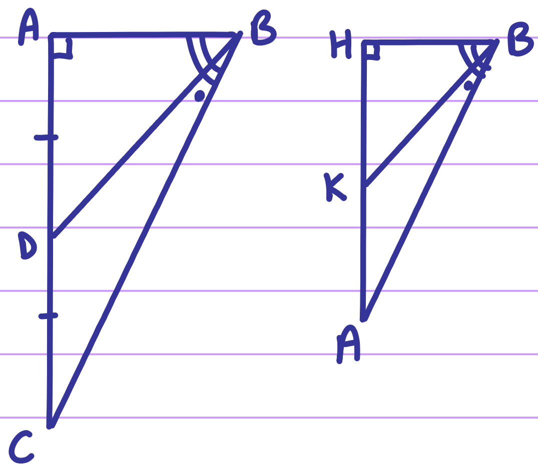
$\therefore \Delta DMB \parallel\!| \Delta BMP$ (SAS similarity test)

(iv) $\angle DBM = \angle BPM$ (corresponding angles in ||| Δs are equal)

$\angle BPM = \angle ABK$ (alternate angles on parallel lines AB & PM =)

$\therefore \angle DBM = \angle ABK$

(v) In ΔABD & ΔHBK



$\angle DAB = \angle KHB = 90°$ (given)

$\angle ABD = \angle ABC - \angle DBC$

$= \angle HBA - \angle KBA$

($\angle ABC = \angle HBA$ common & $\angle DBC = \angle KBA$ part iv)

$= \angle HBK$

$\therefore \Delta ABD$ ||| ΔHBK (AA similarity test)

In ΔDCB & ΔKAB

$\angle CBD = \angle ABK$ (part iv)

$\angle DCB = 180 - 90 - \angle ABC$ ($\angle$ sum $\Delta = 180°$)

$= 180 - 90 - \angle HBA$ (common)

$= \angle KAB$ ($\angle$ sum $\Delta = 180°$)

$\therefore \Delta DCB$ ||| ΔKAB (AA similarity test)

$\frac{AD}{HK} = \frac{DB}{KB}$ (corresponding sides in ||| Δs ABD & HBK in same ratio)

$\frac{DB}{KB} = \frac{DC}{KA}$ (corresponding sides in ||| Δs DCB & KAB in same ratio)

$\therefore \frac{AD}{HK} = \frac{DC}{KA} \Rightarrow \frac{AD}{DC} = \frac{HK}{KA}$

$\Rightarrow \therefore \frac{HK}{KA} = 1$ (since AD = DC)

$\therefore HK = KA$